Identificación de coeficientes en tomografía óptica via un modelo de tipo Level-Set.

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Joint work with:

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What is Diffuse Optical Tomography (DOT)?

- DOT is a non-invasive technique that utilize light in the near infrared spectral region to measure the optical properties of physical body.
- The object under study has to be light-transmitting or translucent, so it works best on soft tissues such as breast and brain tissue.
- By monitoring spatial-temporal variations in the light absorption and scattering of the tissue, spatial maps of properties such as total hemoglobin concentration, blood oxygen saturation and scattering can be obtained.
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The mathematical model

A simplified equation to model the light propagation is the following:

$$(DP)\begin{cases} -\nabla\cdot(a(x)\nabla u)+c(x)u=0 & \text{in } \Omega \\ a(x)\dfrac{\partial u}{\partial \nu}=g & \text{on } \Gamma. \end{cases}$$

- u photon density.
- $a(x)$ diffusion coefficient.
- $c(x)$ absorption coefficient.

Forward map

Parameter-to-measurement (forward) map

$$\begin{aligned} F := F_g : D(F) &\rightarrow H^{1/2}(\Gamma) \\ (a,c) &\mapsto h := u|_\Gamma, \end{aligned}$$

- where $u = u(g)$ is the unique solution of (DP) given the boundary data g and the pair (a,c).
- $D(F)$ is the set of piecewise constant functions $(a,c) \in [L^1(\Omega)]^2$ s.t.

$$\underline{a} \leq a(x) \leq \overline{a}, \quad \underline{c} \leq c(x) \leq \overline{c} \quad a.e. \text{ in } \Omega,$$

where $\underline{a}$, $\overline{a}$, $\underline{c}$ and $\overline{c}$ are known non negative real numbers.

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Inverse problem

- Since the optical properties within tissue are determined by the values of the diffusion and absorption coefficients, the problem of interest in DOT is the simultaneous identification of both coefficients from measurements of near-infrared diffusive light along the tissue boundary.
- Given a finite number of measurements h_m, corresponding to inputs $g_m = \frac{\partial u_m}{\partial \nu}$.

Find $(a,c) \in D(F)$ such that

$$F_m(a,c) = h_m, \quad \text{for } m = 1,\ldots,M. \tag{1}$$

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Inverse problem

- Given the nature of the measurements, we can not expect that exact data h_m are available. Instead, one disposes only an approximate measured data h_m^δ satisfying

$$\left\| h_m - h_m^\delta \right\|_{L^2(\Gamma)} \leq \delta, \quad \text{for } m = 1, \ldots, M$$

where $\delta > 0$ is the noise level.

Find $(a, c) \in D(F)$ such that

$$F_m(a, c) = h_m^\delta, \quad \text{for } m = 1, \ldots, M. \tag{2}$$

Level set approach

- Level set functions $\phi^a, \phi^c \in H^1(\Omega)$ are chosen in such a way that discontinuities of the coeficcients (a,c) are located "along" its zero level sets $\Gamma_{\phi^i} := \{x \in \Omega \,|\, \phi^i(x) = 0\}$.

- The diffusion and absorption coefficients can be written as

$$(a,c) \;=\; \big(a^2 + (a^1 - a^2)H(\phi^a), c^2 + (c^1 - c^2)H(\phi^c)\big) \;=:\; P(\phi^a, \phi^c)$$

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Level set regularization

- A natural alternative to obtain stable solutions is to use a least-square approach combined with a Tikhonov-type regularization

$$\mathcal{F}_\alpha(\phi^a,\phi^c) := \sum_{m=1}^{M} \|F_m(P(\phi^a,\phi^c)) - h_m^\delta\|_{L^2(\Gamma)}^2 + \alpha R(\phi^a,\phi^c) \qquad (3)$$

where

$$R(\phi^a,\phi^c) = \|\phi^a - \phi_0^a\|_{H^1(\Omega)}^2 + \|\phi^c - \phi_0^c\|_{H^1(\Omega)}^2 + \beta_a |H(\phi^a)|_{\mathrm{BV}(\Omega)} + \beta_c |H(\phi^c)|_{\mathrm{BV}(\Omega)}$$

- $\alpha > 0$ plays the role of a regularization parameter and β_j are scaling facor.
- The $H^1(\Omega)$ terms act as a control on the size of the norm of the level set function (key role to prove uniqueness of the existence of ϕ^i).
- The $\mathrm{BV}(\Omega)$-seminorm terms penalize the length of the Hausdorff measure of the boundary of the sets $\Gamma_0^{\phi^i}$.

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Continuous operator

For each $\varepsilon > 0$, the smooth approximations

- $H_\varepsilon(t) := \begin{cases} 1 + t/\varepsilon & \text{for } t \in [-\varepsilon, 0] \\ H(t) & \text{for } t \in \mathbb{R} \setminus [-\varepsilon, 0] \end{cases}$
- $P_\varepsilon(\phi^a, \phi^c) \ := \ \big(a^2 + (a^1 - a^2) H_\varepsilon(\phi^a),\ c^2 + (c^1 - c^2) H_\varepsilon(\phi^c)\big)$

The concept of generalized minimizers

- A **vector** $(z^1, z^2, \phi^a, \phi^c) \in [L^\infty(\Omega)]^2 \times [H^1(\Omega)]^2$ is called **admissible** if there exist sequences $\{\phi_k^j\}$ of H^1-functions and a sequence $\{\varepsilon_k\} \in \mathbb{R}^+$ converging to zero such that

$$\lim_{k\to\infty} \|\phi_k^j - \phi^j\|_{L^2(\Omega)} = 0 \quad \text{and} \quad \lim_{k\to\infty} \|H_{\varepsilon_k}(\phi_k^j) - z^j\|_{L^1(\Omega)} = 0 \, .$$

- A **generalized minimizer** of the functional $\mathcal{F}_\alpha$ in (3) is any admissible vector $(z^1, z^2, \phi^a, \phi^c)$ minimizing

$$\hat{\mathcal{F}}_\alpha(z^1, z^2, \phi^a, \phi^c) := \sum_{m=1}^{M} \|F_m(Q(z^1, z^2)) - h_m^\delta\|_{L^2(\Gamma)}^2 + \alpha\rho(z^1, z^2, \phi^a, \phi^c), \tag{4}$$

  - $Q(z^1, z^2) := (a^2 + (a^1 - a^2) z^1,\, c^2 + (c^1 - c^2) z^2)$,
  - $\rho(z^1, z^2, \phi^a, \phi^c) := \inf \left\{ \liminf_{k\to\infty} \sum_{j=1}^{2} \left(\beta_j |H_{\varepsilon_k}(\phi_k^j)|_{BV} + \|\phi_k^j - \phi_0^j\|_{H^1}^2 \right) \right\}$

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Convergence Analysis

Theorem (DC-L-T 2009)

1. **[Well-posedness]** $\hat{\mathcal{F}}_\alpha$ in (4) attains minimizers on the set of admissible vectors.
2. **[Convergence for exact data]** Assume that $h^\delta = h$. For every $\alpha > 0$ denote by $(z^1_\alpha, z^2_\alpha, \phi^a_\alpha, \phi^c_\alpha)$ a minimizer of $\hat{\mathcal{F}}_\alpha$. Then, for every sequence of positive numbers $\{\alpha_k\}$ converging to zero there exists a subsequence, denoted again by $\{\alpha_k\}$, such that $(z^1_{\alpha_k}, z^2_{\alpha_k}, \phi^a_{\alpha_k}, \phi^c_{\alpha_k})$ is strongly convergent in $[L^1(\Omega)]^2 \times [L^2(\Omega)]^2$. Moreover, the limit is a solution of (1).
3. **[Convergence for noisy data]** Let $\alpha = \alpha(\delta)$ be a function satisfying $\lim_{\delta \to 0} \alpha(\delta) = 0$ and $\lim_{\delta \to 0} \delta^2 \alpha(\delta)^{-1} = 0$. Moreover, let $\{\delta_k\}$ be a sequence of positive numbers converging to zero and $\{h^{\delta_k}\}$ be corresponding noisy data. Then, there exists a subsequence, denoted again by $\{\delta_k\}$, and a sequence $\{\alpha_k := \alpha(\delta_k)\}$ such that $(z^1_{\alpha_k}, z^2_{\alpha_k}, \phi^a_{\alpha_k}, \phi^c_{\alpha_k})$ converges in $[L^1(\Omega)]^2 \times [L^2(\Omega)]^2$ to a solution of (2).

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Convergence Analysis

Generalized Meyers' Theorem

Let $\Omega \subset \mathbb{R}^n$, $n \in \{2,3,4\}$, be a connected bounded open set with a Lipschitz boundary Γ and let $(a,c) \in D(F)$. Then, there exists a real number $p_M > 2$ (depending only on Ω, $\underline{a}, \overline{a}, \underline{c}$ and $\overline{c}$) such that the following condition hold for every $p \in (2, p_M)$:
If $g \in W^{1-(1/q),q}(\Gamma)'$, where $q := p/(p-1)$, then the unique solution u of (DP) belongs to $W^{1,p}(\Omega)$.

Level set regularization: numerical realization.

- In this case, the energy functional is:

$$\mathcal{F}_{\alpha,\varepsilon}(\phi^a,\phi^c) := \sum_{m=1}^{M} \|F_m(P_\varepsilon(\phi^a,\phi^c)) - h_m^\delta\|^2_{L^2(\Gamma)} + \alpha R_\varepsilon(\phi^a,\phi^c)$$

where

$$R_\varepsilon(\phi^a,\phi^c) = |H_\varepsilon(\phi^a)|_{\mathrm{BV}(\Omega)} + |H_\varepsilon(\phi^c)|_{\mathrm{BV}(\Omega)} + \|\phi^a - \phi_0^a\|^2_{H^1(\Omega)} + \|\phi^c - \phi_0^c\|^2_{H^1(\Omega)}$$

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1. Given $\alpha, \varepsilon > 0$ and $\phi_0^i \in H^1$, the functional $\mathcal{F}_{\alpha,\varepsilon}$ attains a minimizer on $[H^1(\Omega)]^2$.
2. Let α be given. For each $\varepsilon > 0$ denote by $(\phi^a_{\varepsilon,\alpha}, \phi^c_{\varepsilon,\alpha})$ a minimizer of $\mathcal{F}_{\alpha,\varepsilon}$. There exists a sequence of positive numbers $\{\varepsilon_k\}$ converging to zero such that $(\phi^a_{\varepsilon_k,\alpha}, \phi^c_{\varepsilon_k,\alpha})$ converges strongly in $[L^2(\Omega)]^2$ and the limit is a generalized minimizer of $\mathcal{F}_\alpha$.

- Differently from $\mathcal{F}_\alpha$, the minimizers of $\mathcal{F}_{\alpha,\varepsilon}$ can be computed.
- Derive the first order optimality condition for a minimizer of $\mathcal{F}_{\alpha,\varepsilon}$.

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- First order optimality condition: $\frac{\partial \mathcal{F}_{\alpha,\varepsilon}}{\partial \phi^j}(h) = 0 \quad \forall h \in H^1(\Omega)$.

$$\begin{aligned} \alpha(\Delta - I)(\phi^j - \phi_0^j) &= L^j_{\varepsilon,\alpha}(\phi^a, \phi^c) \quad \text{in } \Omega \\ \frac{\partial}{\partial \nu}(\phi^j - \phi_0^j) &= 0 \quad \text{on } \Gamma. \end{aligned}$$

$$\begin{aligned} L^a_{\varepsilon,\alpha}(\phi^a, \phi^c) = \; & (a^1 - a^2) H'_\varepsilon(\phi^a) \left[\sum_{m=1}^{M} \left(\frac{\partial F_m(P_\varepsilon(\phi^a, \phi^c))}{\partial \phi^a} \right)^* (F_m(P_\varepsilon(\phi^a, \phi^c)) - h_m^\delta) \right] \\ & -\alpha\beta_a \left[H'_\varepsilon(\phi^a) \nabla \cdot \left(\frac{\nabla H_\varepsilon(\phi^a)}{|\nabla H_\varepsilon(\phi^a)|} \right) \right] \end{aligned}$$

Iterative regularization algorithm

1. Evaluate the residual

$$r_{k,m} := F_m(P_\varepsilon(\phi_k^a, \phi_k^c)) - h_m = u_{k,m}{}_{|_\Gamma} - h_m, \qquad m = 1, \dots, M.$$

2. Evaluate $\left(\frac{\partial F_m(P_\varepsilon(\phi_k^a,\phi_k^c))}{\partial a}\right)^*$ and $\left(\frac{\partial F_m(P_\varepsilon(\phi_k^a,\phi_k^c))}{\partial c}\right)^*$ $\quad m = 1, \dots, M.$

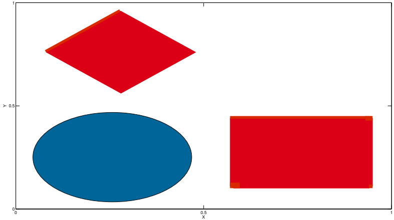
3. Calculate $\delta\phi_k^i$ solutions of the BVP

$$\begin{cases} (\Delta - I)\delta\phi_k^i = L_{\varepsilon,\alpha}^i(\phi_k^a, \phi_k^c) & \text{in } \Omega \\ \frac{\partial \delta\phi_k^i}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases}$$

4. Update the level set functions

$$\phi_{k+1}^a = \phi_k^a + \delta\phi_k^a \quad \text{and} \quad \phi_{k+1}^c = \phi_k^c + \delta\phi_k^c$$

Numerical Examples



$$a^*(x) = \begin{cases} 10\,, & \text{inside blue inclusion} \\ 1\,, & \text{elsewhere} \end{cases}\,, \quad c^*(x) = \begin{cases} 10\,, & \text{inside red inclusion} \\ 1\,, & \text{elsewhere.} \end{cases}$$

Four ($M = 4$) distinct functions g_m, each one supported at each side of Γ.

Identification of the absorption coefficient $c(x)$

a^* is assumed to be exactly known

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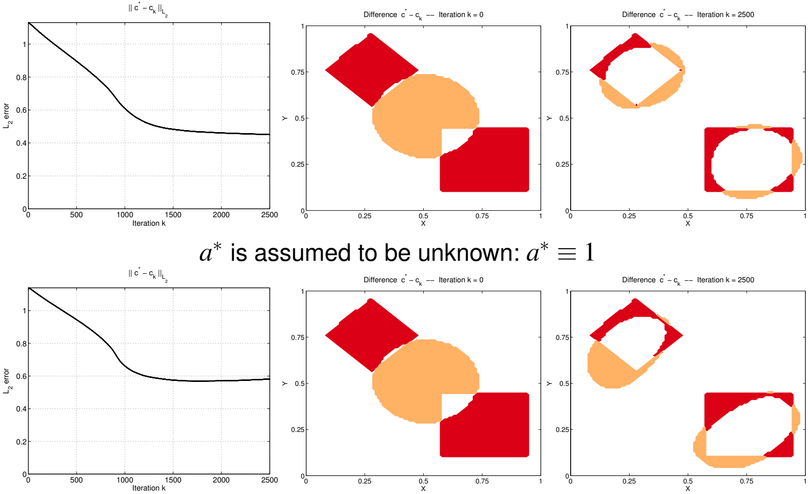
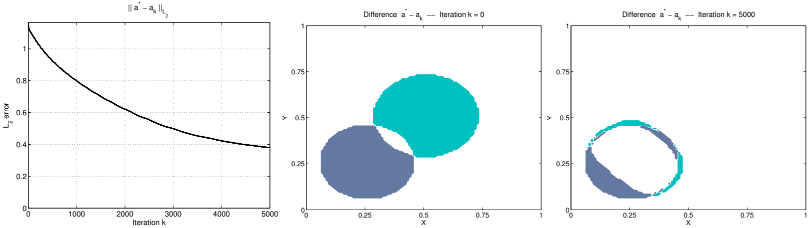


Identification of the diffusion coefficient $a(x)$

c^* is assumed to be exactly known



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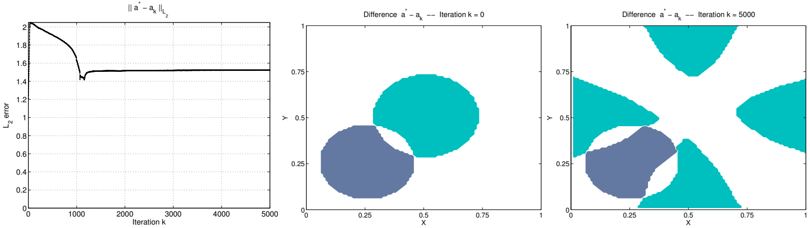

Split strategy

- Some facts to take into account:
  1. The method for identifying c^* performs well, even if a good approximation of a^* is not known.
  2. On the other hand, the method may generate a sequence a^k that does not approximate a^* if $\|c^k - c^*\|$ is large.
  3. For simultaneous identification of (a^*, c^*) we observed that the error $\|c^k - c^*\|$ decreases from the very first iteration. However, the error $\|a^k - a^*\|$ only starts improving when $\|c^k - c^*\|$ is sufficiently small.
- Split strategy:
  1. Set $a^k(x) \equiv 1$ and iterate w.r.t. c^k until the sequenece c^k stagnates ($\|c^k - c^*\|$ is small).
  2. Set $a^k(x) \equiv a^{k_1}$ and iterate w.r.t. a^k until the sequenece a^k stagnates ($\|a^k - a^*\|$ is small).
  3. Each iteration step consist in one iteration w.r.t. c^k and two iterations w.r.t a^k.

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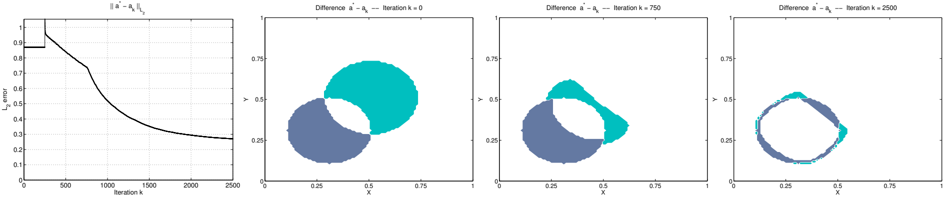
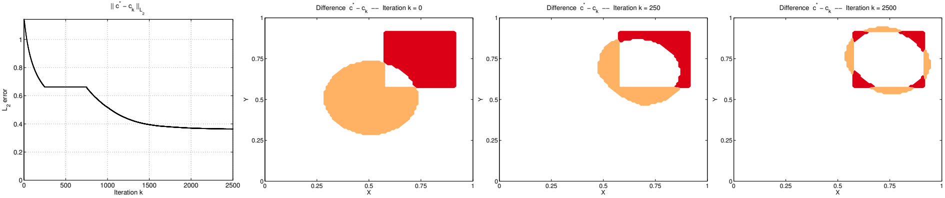
  1. The method for identifying c^* performs well, even if a good approximation of a^* is not known.
  2. On the other hand, the method may generate a sequence a^k that does not approximate a^* if $\|c^k - c^*\|$ is large.
  3. For simultaneous identification of (a^*, c^*) we observed that the error $\|c^k - c^*\|$ decreases from the very first iteration. However, the error $\|a^k - a^*\|$ only starts improving when $\|c^k - c^*\|$ is sufficiently small.
- Split strategy:
  1. Set $a^k(x) \equiv 1$ and iterate w.r.t. c^k until the sequenece c^k stagnates ($\|c^k - c^*\|$ is small).
  2. Set $a^k(x) \equiv a^{k_1}$ and iterate w.r.t. a^k until the sequenece a^k stagnates ($\|a^k - a^*\|$ is small).
  3. Each iteration step consist in one iteration w.r.t. c^k and two iterations w.r.t a^k.

Identification of both coefficients: example 1

Diffusion coeficcient $a(x)$

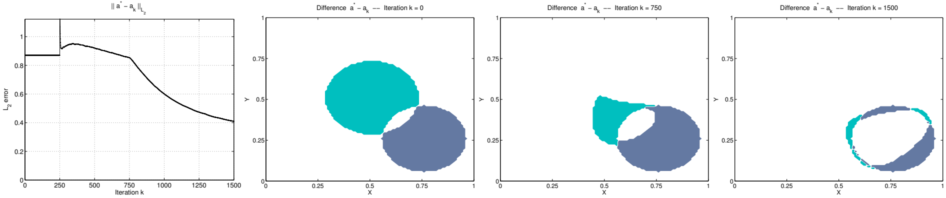


Absorption coefficient $c(x)$

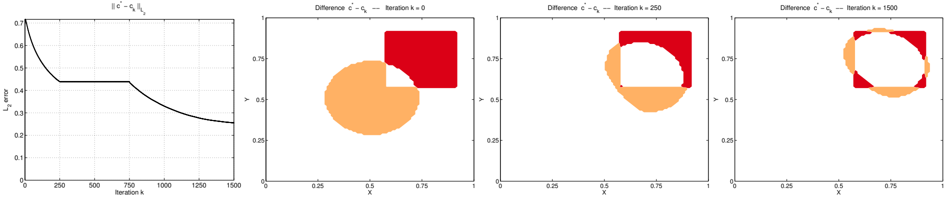


Identification of both coefficients: example 2

Diffusion coeficcient $a(x)$



Absorption coefficient $c(x)$



Some comments

- We developed a level set approach for simultaneous reconstruction of the piecewise constant coefficients (a, c) from a finite set of boundary measurements of optical tomography in the diffusive regime.
- We proved that the forward map F is continuous in the L^1-topology. Hence, by previous results, the presented level set approach is a regularization method.
- We proposed a split strategy for the simultaneous identification. Such strategy produces very good results when a^* and c^* have no crossing supports.
- The strategy reduces significatively the numerical computational time.

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¡ Muchas gracias !